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Research Article

A HIGHER ORDER TRIGONOMETRICALLY-FITTED METHOD FOR SECOND ORDER NONLINEAR PERIODIC PROBLEMS

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ABSTRACT

This paper presents a higher order, block implicit, four step method with trigonometric coefficients constructed via multistep collocation technique. The stability properties of the method are discussed. Numerical results obtained disclose that the new method is suitable for the integration of second order nonlinear periodic problems.

Keywords: Collocation Technique; Nonlinear Periodic Problems; Trigonometrically-Fitted.

Mathematics Subject Classification: 65L05, 65L06

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1. INTRODUCTION

This paper proposed and applied a Block Implicit Trigonometrically-Fitted Method (BITM) of higher order to solve second order Initial Value Problems (IVPs) of the form

$$\left.\begin{aligned} y''(x) = \psi(x, y(x))\ , \qquad x \in [x_0\,, x_N] \\ y(x_0) = y_0,\ \ y'(x_0) = y_0' \end{aligned}\right\} \qquad (1)$$

for which the solutions are periodic in nature, where $\psi: [x_0\,, x_N\,] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is sufficiently differentiable, satisfies the conditions of existence and uniqueness of solution (See [1] and [2]). Equation (1) is frequently encountered in pure and applied mathematics and in several area of applied science and engineering, such as mechanics, physics, molecular biology and control theory. In applications, most equations in the form of equation (1) do not possess exact solutions or the exact solutions are not easily obtained. Thus, numerical methods become necessary for solving equation (1).

A number of numerical methods built on traditional Runge-Kutta (RK) methods, Linear Multistep Methods (LMM), Boundary Value Methods (BVM), Exponential Fitted (EF) methods and Trigonometrically-Fitted methods have been discussed and investigated in literatures for solving equation (1) (see [3]-[16]) and are referenced therein. It turns out that some of these methods are of low order, some have many numbers of function evaluation particularly those executed in predictor-corrector mode, the hybrid methods are compounded with the need to develop predictors for the evaluation of the correctors at the off step points.

Numerical methods for solving equation (1) which involves higher-order derivatives have been discussed in [11] and [17]. Ehigie [18] emphasized that the use of higher derivatives in formulations of numerical schemes can reduce error constant more rapidly than increasing the number of steps in a multistep method. [14] averred that methods with higher derivatives are more often favourable, since meeting stability condition for multistep methods is often demanding.

It is against this background that, a block trigonometric method with fewer function evaluation that is self-starting is developed for solving equation (1), which is first transformed to the system of first order IVPs of the form

$$y' = f(x, y), \;\; y(x_0) = y_0 \tag{2}$$

before its implementation. The BITM is applied on the partition $x_0 < x_1 < x_2 < \cdots < x_N$ over non-overlapping subintervals without the need for predictors and is formulated by combining the method

$$y_{n+4} = \alpha_0(\sin u, \cos u)y_n + \alpha_1(\sin u, \cos u)y_{n+1} + \alpha_2(\sin u, \cos u)y_{n+2} + \alpha_3(\sin u, \cos u)y_{n+3} + h(\beta_0(\sin u, \cos u)f_n + \beta_1(\sin u, \cos u)f_{n+1} + \beta_2(\sin u, \cos u)f_{n+2} + \beta_3(\sin u, \cos u)f_{n+3} + \beta_4(\sin u, \cos u)f_{n+4}) + h^2\gamma_4(\sin u, \cos u)g_{n+4} \tag{3}$$

with complementary methods given by

$$h^2 g_{n+1} = \overline{\alpha_{0,1}}(\sin u, \cos u) y_n + \overline{\alpha_{1,1}}(\sin u, \cos u) y_{n+1} + \overline{\alpha_{2,1}}(\sin u, \cos u) y_{n+2} + \overline{\alpha_{3,1}}(\sin u, \cos u) y_{n+3} + h\big(\overline{\beta_{0,1}}(\sin u, \cos u) f_n + \overline{\beta_{1,1}}(\sin u, \cos u) f_{n+1} + \overline{\beta_{2,1}}(\sin u, \cos u) f_{n+2} + \overline{\beta_{3,1}}(\sin u, \cos u) f_{n+3} + \overline{\beta_{4,1}}(\sin u, \cos u) f_{n+3}\big) + h^2 \overline{\gamma_{4,1}}(\sin u, \cos u) g_{n+4} \quad (4)$$

$$h^2 g_{n+2} = \overline{\alpha_{0,2}}(\sin u, \cos u) y_n + \overline{\alpha_{1,2}}(\sin u, \cos u) y_{n+1} + \overline{\alpha_{2,2}}(\sin u, \cos u) y_{n+2} + \overline{\alpha_{3,2}}(\sin u, \cos u) y_{n+2} + h\big(\overline{\beta_{0,2}}(\sin u, \cos u) f_n + \overline{\beta_{1,2}}(\sin u, \cos u) f_{n+1} + \overline{\beta_{2,2}}(\sin u, \cos u) f_{n+2} + \overline{\beta_{3,2}}(\sin u, \cos u) f_{n+3} + \overline{\beta_{4,2}}(\sin u, \cos u) f_{n+3}\big) + h^2 \overline{\gamma_{4,2}}(\sin u, \cos u) g_{n+4} \quad (5)$$

$$h^2 g_{n+3} = \overline{\alpha_{0,3}}(\sin u, \cos u) y_n + \overline{\alpha_{1,3}}(\sin u, \cos u) y_{n+1} + \overline{\alpha_{2,3}}(\sin u, \cos u) y_{n+2} + \overline{\alpha_{3,3}}(\sin u, \cos u) y_{n+2} + h\big(\overline{\beta_{0,3}}(\sin u, \cos u) f_n + \overline{\beta_{1,3}}(\sin u, \cos u) f_{n+1} + \overline{\beta_{2,3}}(\sin u, \cos u) f_{n+2} + \overline{\beta_{3,3}}(\sin u, \cos u) f_{n+3} + \overline{\beta_{4,3}}(\sin u, \cos u) f_{n+3}\big) + h^2 \overline{\gamma_{4,3}}(\sin u, \cos u) g_{n+4} \quad (6)$$

where $\alpha_i, \overline{\alpha_{j,\iota}}, \beta_r, \overline{\beta_{r,\iota}}, \gamma_k, \overline{\gamma_{k,\iota}}$, $j = \{0,1,2,3\}, i = \{1,2,3\}, r = \{0,1,2,3,4\}$ and $k = 4$ are coefficients to be ascertained distinctively via multistep collocation method. These coefficients are selected so that BITM integrates equation (1) exactly, where the solutions are member of any linear combination of the function $\{1, x, x^2, x^3, x^4, x^5, x^6, x^7\} \cup \{\sin(\omega x), \cos(\omega x)\}$, and $\omega = uh$.

2. DEVELOPMENT OF BITM

Our starting point in this section is to construct the continuous approximation for BITM via multistep collocation technique which has the form

$$\tau(x) = \sum_{j=0}^{3} \alpha_j(x,u)\, y_{n+j} + h \sum_{j=0}^{4} \beta_j(x,u)\, f_{n+j} + h^2 \gamma_4(x,u) g_{n+4} \quad (7)$$

2.1 Continuous Approximation for the BITM

We assume that the exact solution $y(x)$ is locally approximated by seeking the solution of the form

$$\tau(x) = \sum_{j=0}^{7} a_j\, x^j + a_8 \sin(\omega x) + a_9 \cos(\omega x)$$

We specifically demand that the following 10 system of equations be satisfied

$$\tau(x_{n+j}) = y_{n+j} \quad , \quad j = 0,1,2,3 \qquad (8),$$

$$\tau'(x)|_{x=x_{n+j}} = f_{n+j} \quad , \quad j = 0,1,2,3,4 \qquad (9),$$

$$\tau''(x)|_{x=x_{n+4}} = g_{n+4} \qquad (10).$$

We now state the theorem that aids the construction of the continuous method as follows:

Theorem 1

Let $\tau(x)$ satisfies the system of 10 equations obtained in equations (8)-(10), then the continuous approximation used to obtain equation (3) and equations (4)-(6) are given by

$$\tau(x) = \Theta^T(\Omega^{-1})^T\sigma(x)$$

$$\tau''(x) = \frac{d^2}{dx^2}(\Theta^T(\Omega^{-1})^T\sigma(x))$$

where Ω is a 10×10 matrix given by

$$\Omega = \begin{bmatrix} \sigma_0(x_n) & \sigma_1(x_n) & \sigma_2(x_n) & \cdots & \sigma_9(x_n) \\ \sigma_0(x_{n+1}) & \sigma_1(x_{n+1}) & \sigma_2(x_{n+1}) & \cdots & \sigma_9(x_{n+1}) \\ \sigma_0(x_{n+2}) & \sigma_1(x_{n+2}) & \sigma_2(x_{n+2}) & \cdots & \sigma_9(x_{n+2}) \\ \sigma_0(x_{n+3}) & \sigma_1(x_{n+3}) & \sigma_2(x_{n+3}) & \cdots & \sigma_9(x_{n+3}) \\ \sigma_0'(x_n) & \sigma_1'(x_n) & \sigma_2'(x_n) & \cdots & \sigma_9'(x_n) \\ \sigma_0'(x_{n+1}) & \sigma_1'(x_{n+1}) & \sigma_2'(x_{n+1}) & \cdots & \sigma_9'(x_{n+1}) \\ \sigma_0'(x_{n+2}) & \sigma_1'(x_{n+2}) & \sigma_2'(x_{n+2}) & \cdots & \sigma_9'(x_{n+2}) \\ \sigma_0'(x_{n+3}) & \sigma_1'(x_{n+3}) & \sigma_2'(x_{n+3}) & \cdots & \sigma_9'(x_{n+3}) \\ \sigma_0'(x_{n+4}) & \sigma_1'(x_{n+4}) & \sigma_2'(x_{n+4}) & \cdots & \sigma_9'(x_{n+4}) \\ \sigma_0''(x_{n+4}) & \sigma_1''(x_{n+4}) & \sigma_2''(x_{n+4}) & \cdots & \sigma_9''(x_{n+4}) \end{bmatrix},$$

Θ and $\sigma(x)$ are vectors defined by

$\Theta = (y_n, y_{n+1}, y_{n+2}, y_{n+3}, f_n, f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}, g_{n+4})^T$ and

$\sigma(x) = (\sigma_0(x), \sigma_1(x), \sigma_2(x), \sigma_3(x), \sigma_4(x), \sigma_5(x), \sigma_6(x), \sigma_7(x), \sigma_8(x), \sigma_9(x))^T$ respectively, and T is the transpose.

Proof

To solve the system of equations (8)-(10), we required that equation (7) be defined by the assumed basis function as follows

$$\alpha_j(x,u) = \sum_{i=0}^{9} \alpha_{i,j}(x,u)\,\sigma_i(x) \quad j = 0,1,2,3 \tag{13}$$

$$h\beta_j(x,u) = \sum_{i=0}^{9} h\beta_{i,j}(x,u)\,\sigma_i(x) \quad j = 0,1,2,3,4 \tag{14}$$

$$h^2\gamma_4(x,u) = \sum_{i=0}^{9} h^2\gamma_{i,4}(x,u)\,\sigma_i(x) \tag{15}$$

Substituting equations (13)-(15) into equation (7) yield

$$\tau(x) = \sum_{i=0}^{9}\left\{\sum_{j=0}^{3} \alpha_{i,j}(x,u)\,y_{n+j} + h\sum_{j=0}^{4} \beta_{i,j}(x,u)\,f_{n+j} + h^2\gamma_{i,4}(x,u)g_{n+4}\right\}\sigma_i(x) \tag{16}$$

Letting

$$\Delta_i = \sum_{j=0}^{3} \alpha_{i,j}(x,u)\,y_{n+j} + h\sum_{j=0}^{4} \beta_{i,j}(x,u)\,f_{n+j} + h^2\gamma_{i,4}(x,u)g_{n+4}$$

equation (16) becomes

$$\tau(x) = \sum_{i=0}^{9} \Delta_i\,\sigma_i(x) \tag{17}$$

Imposing the conditions in equations (8)-(10) on equation (17), we obtain a system of 10 equations which is expressed as $\Omega\Delta = \Theta$ where $\Delta = (\Delta_0, \Delta_1, \Delta_2 \cdots, \Delta_9)^T$ is a vector form of 10 undetermined coefficients that are ascertained by applying matrix inversion method since Ω is a nonsingular matrix to obtain

$$\Delta = \Omega^{-1}\Theta \tag{18}$$

Re-writing equation (17) in vector form gives

$$\tau(x) = \Delta^T\sigma(x) \tag{19}$$

It follows from equations (18) and (19) that

$$\tau(x) = \Theta^T(\Omega^{-1})^T\sigma(x) \tag{20}$$

Differentiating equation (20) with respect to x twice gives

$$\tau''(x) = \frac{d^2}{dx^2}\left(\Theta^T(\Omega^{-1})^T\sigma(x)\right) \tag{21}$$

We emphasis that equations (20) and (21) are the continuous methods given by

$$\tau(x) = \sum_{j=0}^{3} \alpha_j(x,u)\,y_{n+j} + h\sum_{j=0}^{4} \beta_j(x,u)\,f_{n+j} + h^2\gamma_4(x,u)g_{n+4} \tag{22}$$

$$\tau''(x) = \frac{1}{h^2}\sum_{j=0}^{3}\overline{\alpha_{j,\iota}}(x,u)\,y_{n+j} + \frac{1}{h}\sum_{j=0}^{4}\overline{\beta_{j,\iota}}(x,u)\,f_{n+j} + \overline{\gamma_{4,\iota}}(x,u)g_{n+4} \tag{23}$$

from which the main method given by equation (3) and complementary methods given by equations (4)-(6) are obtained respectively.

2.2 Specification of BITM

The main method in equation (3) is obtained by evaluating equation (22) at $= x_{n+4}$, while the 3 complementary methods in equations (4)-(6) are obtained by evaluating equation (23) at $x = x_{n+i}\,, i = 1{,}2{,}3$ which are respectively written in compact form as

$$y_{n+4} = \sum_{j=0}^{3}\alpha_j(\sin u\,,\cos u)\,y_{n+j} + h\sum_{j=0}^{4}\beta_j(\sin u\,,\cos u)\,f_{n+j} + h^2\gamma_4(\sin u\,,\cos u)g_{n+4} \tag{24}$$

$$h^2 g_{n+i} = \sum_{j=0}^{3}\overline{\alpha_{j,\iota}}(\sin u\,,\cos u)\,y_{n+j} + h\sum_{j=0}^{4}\overline{\beta_{j,\iota}}(\sin u\,,\cos u)\,f_{n+j} + h^2\overline{\gamma_{4,\iota}}(\sin u\,,\cos u)g_{n+4} \tag{25}$$

The coefficients of equation (24) are given in equation (26) below

$$\begin{aligned}\alpha_0 = {} & \Big(\big(-396\,u^2\cos(u)^3 + (13344\,u^2 - 61560)\cos(u)^2 + (36990\,u^2 - 12960)\cos(u) + 25032\,u^2 \\ & + 74520\big)\sin(u) + u\,\big(54\cos(u)^4 u^2 - 1368\,u^2\cos(u)^3 - 873\cos(u)^4 - 3618\cos(u)^2 u^2 \\ & + 49808\cos(u)^3 - 216\cos(u)\,u^2 + 91179\cos(u)^2 + 1683\,u^2 - 49200\cos(u) - 90914\big)\Big) \Big/ \Big(\big(\\ & -6012\,u^2\cos(u)^3 + (-8640\,u^2 - 104760)\cos(u)^2 + (75786\,u^2 - 108000)\cos(u) + 90696\,u^2 \\ & + 212760\big)\sin(u) + u\,\big(846\cos(u)^4 u^2 + 4968\,u^2\cos(u)^3 - 10523\cos(u)^4 + 2718\cos(u)^2 u^2 \\ & + 58896\cos(u)^3 - 3384\cos(u)\,u^2 + 268125\cos(u)^2 - 1683\,u^2 - 42736\cos(u) - 273762\big)\Big)\end{aligned}$$

$$\begin{aligned}\alpha_1 = {} & \Big(\big((-24480\,u^2 + 123120)\cos(u)^3 + (-65664\,u^2 - 17280)\cos(u)^2 + (6288\,u^2 - 244080)\cos(u) \\ & + 48576\,u^2 + 138240\big)\sin(u) + 16\,u\,\big(144\cos(u)^4 u^2 + 432\,u^2\cos(u)^3 - 5786\cos(u)^4 \\ & - 468\cos(u)^2 u^2 - 9900\cos(u)^3 - 576\cos(u)\,u^2 + 15801\cos(u)^2 + 153\,u^2 + 10532\cos(u) \\ & - 10647\big)\Big) \Big/ \Big(\big(-6012\,u^2\cos(u)^3 + (-8640\,u^2 - 104760)\cos(u)^2 + (75786\,u^2 - 108000)\cos(u) \\ & + 90696\,u^2 + 212760\big)\sin(u) + u\,\big(846\cos(u)^4 u^2 + 4968\,u^2\cos(u)^3 - 10523\cos(u)^4 \\ & + 2718\cos(u)^2 u^2 + 58896\cos(u)^3 - 3384\cos(u)\,u^2 + 268125\cos(u)^2 - 1683\,u^2 - 42736\cos(u) \\ & - 273762\big)\Big)\end{aligned}$$

$$\begin{aligned}\alpha_2 = {} & \Big(\big((-19440\,u^2 + 86400)\cos(u)^3 + (-121824\,u^2 + 233280)\cos(u)^2 + (-133812\,u^2 \\ & - 181440)\cos(u) - 31104\,u^2 - 138240\big)\sin(u) + 54\,u\,\big(36\cos(u)^4 u^2 + 288\,u^2\cos(u)^3 \\ & - 1259\cos(u)^4 + 288\cos(u)^2 u^2 - 6080\cos(u)^3 - 144\cos(u)\,u^2 - 1941\cos(u)^2 - 153\,u^2 \\ & + 6368\cos(u) + 2912\big)\Big) \Big/ \Big(\big(-6012\,u^2\cos(u)^3 + (-8640\,u^2 - 104760)\cos(u)^2 + (75786\,u^2 \\ & - 108000)\cos(u) + 90696\,u^2 + 212760\big)\sin(u) + u\,\big(846\cos(u)^4 u^2 + 4968\,u^2\cos(u)^3 \\ & - 10523\cos(u)^4 + 2718\cos(u)^2 u^2 + 58896\cos(u)^3 - 3384\cos(u)\,u^2 + 268125\cos(u)^2 - 1683\,u^2 \\ & - 42736\cos(u) - 273762\big)\Big)\end{aligned}$$

$$\alpha_3 = \Big(\big((38304\,u^2 - 209520)\cos(u)^3 + (165504\,u^2 - 259200)\cos(u)^2 + (166320\,u^2 + 330480)\cos(u) + 48192\,u^2 + 138240\big)\sin(u) - 16\,u\,\big(216\cos(u)^4 u^2 + 1008\,u^2\cos(u)^3 - 9432\cos(u)^4 + 108\cos(u)^2 u^2 - 30988\cos(u)^3 - 864\cos(u)\,u^2 - 1809\cos(u)^2 - 153\,u^2 + 31620\cos(u) + 10609\big)\Big) \Big/ \Big(\big(-6012\,u^2\cos(u)^3 + (-8640\,u^2 - 104760)\cos(u)^2 + (75786\,u^2 - 108000)\cos(u) + 90696\,u^2 + 212760\big)\sin(u) + u\,\big(846\cos(u)^4 u^2 + 4968\,u^2\cos(u)^3 - 10523\cos(u)^4 + 2718\cos(u)^2 u^2 + 58896\cos(u)^3 - 3384\cos(u)\,u^2 + 268125\cos(u)^2 - 1683\,u^2 - 42736\cos(u) - 273762\big)\Big)$$

$$\beta_0 = \Big(9\,h\,\big(6\,u^2\cos(u)^3 + 472\cos(u)^2 u^2 - 97\cos(u)^3 + 1173\cos(u)\,u^2 - 1944\cos(u)^2 + 764\,u^2 - 231\cos(u) + 2272\big)\sin(u) + 3\,h\,u\,\big(-144\,u^2\cos(u)^3 + 132\cos(u)^4 - 324\cos(u)^2 u^2 + 5116\cos(u)^3 + 8145\cos(u)^2 + 153\,u^2 - 5076\cos(u) - 8317\big)\Big) \Big/ \Big(\big(-6012\,u^2\cos(u)^3 + (-8640\,u^2 - 104760)\cos(u)^2 + (75786\,u^2 - 108000)\cos(u) + 90696\,u^2 + 212760\big)\sin(u) + u\,\big(846\cos(u)^4 u^2 + 4968\,u^2\cos(u)^3 - 10523\cos(u)^4 + 2718\cos(u)^2 u^2 + 58896\cos(u)^3 - 3384\cos(u)\,u^2 + 268125\cos(u)^2 - 1683\,u^2 - 42736\cos(u) - 273762\big)\Big)$$

$$\beta_1 = \Big(-16\,h\,\big(558\,u^2\cos(u)^3 - 2627\cos(u)^3 - 3924\cos(u)\,u^2 + 6696\cos(u)^2 - 3564\,u^2 + 6459\cos(u) - 10528\big)\sin(u) + 24\,h\,u\,\big(36\cos(u)^4 u^2 - 1371\cos(u)^4 - 360\cos(u)^2 u^2 + 1176\cos(u)^3 - 144\cos(u)\,u^2 + 9773\cos(u)^2 + 153\,u^2 - 1000\cos(u) - 8578\big)\Big) \Big/ \Big(\big(-6012\,u^2\cos(u)^3 + (-8640\,u^2 - 104760)\cos(u)^2 + (75786\,u^2 - 108000)\cos(u) + 90696\,u^2 + 212760\big)\sin(u) + u\,\big(846\cos(u)^4 u^2 + 4968\,u^2\cos(u)^3 - 10523\cos(u)^4 + 2718\cos(u)^2 u^2 + 58896\cos(u)^3 - 3384\cos(u)\,u^2 + 268125\cos(u)^2 - 1683\,u^2 - 42736\cos(u) - 273762\big)\Big)$$

$$\beta_2 = \Big(-108\,h\,\big(378\,u^2\cos(u)^3 + 1272\cos(u)^2 u^2 - 1859\cos(u)^3 + 417\cos(u)\,u^2 - 648\cos(u)^2 - 492\,u^2 + 3723\cos(u) - 1216\big)\sin(u) + 108\,h\,u\,(\cos(u) - 1)\,\big(36\,u^2\cos(u)^3 + 180\cos(u)^2 u^2 - 1409\cos(u)^3 + 144\cos(u)\,u^2 - 4605\cos(u)^2 - 1833\cos(u) + 1547\big)\Big) \Big/ \Big(\big(-6012\,u^2\cos(u)^3 + (-8640\,u^2 - 104760)\cos(u)^2 + (75786\,u^2 - 108000)\cos(u) + 90696\,u^2 + 212760\big)\sin(u) + u\,\big(846\cos(u)^4 u^2 + 4968\,u^2\cos(u)^3 - 10523\cos(u)^4 + 2718\cos(u)^2 u^2 + 58896\cos(u)^3 - 3384\cos(u)\,u^2 + 268125\cos(u)^2 - 1683\,u^2 - 42736\cos(u) - 273762\big)\Big)$$

$$\beta_3 = \Big(-144\,h\,\big(174\,u^2\cos(u)^3 + 656\cos(u)^2 u^2 - 703\cos(u)^3 + 66\cos(u)\,u^2 - 216\cos(u)^2 - 476\,u^2 + 1911\cos(u) - 992\big)\sin(u) + 24\,h\,u\,\big(108\cos(u)^4 u^2 + 576\,u^2\cos(u)^3 - 3513\cos(u)^4 + 216\cos(u)^2 u^2 - 8008\cos(u)^3 - 432\cos(u)\,u^2 + 10467\cos(u)^2 - 153\,u^2 + 8952\cos(u) - 7898\big)\Big) \Big/ \Big(\big(-6012\,u^2\cos(u)^3 + (-8640\,u^2 - 104760)\cos(u)^2 + (75786\,u^2 - 108000)\cos(u) + 90696\,u^2 + 212760\big)\sin(u) + u\,\big(846\cos(u)^4 u^2 + 4968\,u^2\cos(u)^3 - 10523\cos(u)^4 + 2718\cos(u)^2 u^2 + 58896\cos(u)^3 - 3384\cos(u)\,u^2 + 268125\cos(u)^2 - 1683\,u^2 - 42736\cos(u) - 273762\big)\Big)$$

$$\beta_4 = \Big(h\,\big(-846\,u^2\cos(u)^3 + 5832\cos(u)^2 u^2 - 10523\cos(u)^3 + 46719\cos(u)\,u^2 - 67176\cos(u)^2 + 46260\,u^2 - 33789\cos(u) + 111488\big)\sin(u) + 3\,h\,u\,\big(72\cos(u)^4 u^2 + 432\,u^2\cos(u)^3 + 904\cos(u)^4 + 252\cos(u)^2 u^2 + 18996\cos(u)^3 - 288\cos(u)\,u^2 + 44303\cos(u)^2 - 153\,u^2 - 16700\cos(u) - 47503\big)\Big) \Big/ \Big(\big(-6012\,u^2\cos(u)^3 + (-8640\,u^2 - 104760)\cos(u)^2 + (75786\,u^2 - 108000)\cos(u) + 90696\,u^2 + 212760\big)\sin(u) + u\,\big(846\cos(u)^4 u^2 + 4968\,u^2\cos(u)^3 - 10523\cos(u)^4 + 2718\cos(u)^2 u^2 + 58896\cos(u)^3 - 3384\cos(u)\,u^2 + 268125\cos(u)^2 - 1683\,u^2 - 42736\cos(u) - 273762\big)\Big)$$

$$\gamma_4 = \Big(-12\,h^2 \big(18\,u^2 \cos(u)^3 + 216 \cos(u)^2 u^2 - 275 \cos(u)^3 + 729 \cos(u)\, u^2 - 1080 \cos(u)^2 + 612\,u^2 - 165 \cos(u) + 1520\big) \sin(u) - 36\,h^2 u \big(47 \cos(u)^4 + 372 \cos(u)^3 + 572 \cos(u)^2 - 348 \cos(u) - 643\big)\Big) \Big/ \Big(\big(-6012\,u^2 \cos(u)^3 + (-8640\,u^2 - 104760) \cos(u)^2 + (75786\,u^2 - 108000) \cos(u) + 90696\,u^2 + 212760\big) \sin(u) + u \big(846 \cos(u)^4 u^2 + 4968\,u^2 \cos(u)^3 - 10523 \cos(u)^4 + 2718 \cos(u)^2 u^2 + 58896 \cos(u)^3 - 3384 \cos(u)\, u^2 + 268125 \cos(u)^2 - 1683\,u^2 - 42736 \cos(u) - 273762\big)\Big) \qquad (26)$$

To avoid substantial losses of accuracy when evaluating the coefficients that may occur when h is small, the use of power series expansion of $\alpha_j, \beta_j, \gamma_4, \overline{\alpha_{J,\iota}}, \overline{\beta_{J,\iota}}$ and $\overline{\gamma_{4,\iota}}$ is preferable ([13] and [19]). The converted coefficients of the main method of BITM in power series form is given in equation (27).

$$\alpha_0 = \frac{53}{485} + \frac{263968}{116436375} u^2 + \frac{8803197808}{121131671821875} u^4 + \frac{387960688124216}{2035648027884564843 75} u^6 + O(u^8)$$

$$\alpha_1 = \frac{512}{485} - \frac{256512}{12937375} u^2 - \frac{1135105024}{4486358215625} u^4 - \frac{2501866427264}{1077062448616171875} u^6 + O(u^8)$$

$$\alpha_2 = \frac{432}{485} - \frac{332832}{12937375} u^2 - \frac{2495427664}{4486358215625} u^4 - \frac{6704575126856}{579956703101015625} u^6 + O(u^8)$$

$$\alpha_3 = -\frac{512}{485} + \frac{5040128}{116436375} u^2 + \frac{89221184768}{121131671821875} u^4 + \frac{2438197936155136}{203564802788456484375} u^6 + O(u^8)$$

$$\beta_0 = \frac{12}{485} + \frac{30464}{38812125} u^2 + \frac{906424784}{40377223940625} u^4 + \frac{2913213886336}{5219610327909140625} u^6 + O(u^8)$$

$$\beta_1 = \frac{256}{485} - \frac{347264}{116436375} u^2 + \frac{4240004416}{121131671821875} u^4 + \frac{69504995396576}{29080686112636640625} u^6 + O(u^8)$$

$$\beta_2 = \frac{864}{485} - \frac{43584}{1176125} u^2 - \frac{230051168}{407850746875} u^4 - \frac{5318766217936}{685403376392109375} u^6 + O(u^8)$$

$$\beta_3 = \frac{768}{485} - \frac{843904}{38812125} u^2 - \frac{15442335424}{40377223940625} u^4 - \frac{33843479340896}{5219610327909140625} u^6 + O(u^8)$$

$$\beta_4 = \frac{40}{97} + \frac{1024}{423405} u^2 + \frac{19746544}{440478806625} u^4 + \frac{7758660544}{9613449954590625} u^6 + O(u^8)$$

$$\gamma_4 = -\frac{24}{485} - \frac{29888}{38812125} u^2 - \frac{582703328}{40377223940625} u^4 - \frac{17837298594256}{678549342628188281 25} u^6 + O(u^8) \qquad (27)$$

It is also interesting to note that as $u \longrightarrow 0$ in the power series expansion of the parameter $\alpha_j, \beta_j, \gamma_4, \overline{\alpha_{J,\iota}}, \overline{\beta_{J,\iota}}$ and $\overline{\gamma_{4,\iota}}$, methods based on polynomial basis are recovered ([13]).

We also remark that the coefficients of the 3 complementary methods in equations (25) are in trigonometric form and are omitted together with their equivalent power series for brevity

2.3 Convergence of BITM

The convergence of BITM is in the spirit of [20], [21] and [22].

Theorem 2

Let $\overline{\mathrm{Y}}$ be an approximation of the solution vector Y for the system obtained from BITM given by equations (23) and (24). If $e_n = |y(x_n) - y_n|$, where the exact solution is several times differentiable on $[x_0, x_N]$ and if $\|\mathrm{E}\| = \|\overline{\mathrm{Y}} - \mathrm{Y}\|$, then for sufficiently small h, the BITM is a 9th order convergent method. In other words, $\|\mathrm{E}\| = O(h^9)$.

Proof

The proof is similar to the one in [22].

3. ANALYSIS OF BITM

3.1 Local Truncation Errors of BITM

Theorem 3

The BITM has a local truncation error (LTE) of $C_{10}h^{10}\left(\omega^2 y^{(8)}(x_n) + y^{(10)}(x_n)\right) + O\left(h^{(11)}\right)$

Proof:

Consider the Taylor series expansion of the following

$y_{n+j}, y(x_n + jh), y'_{n+j}, y'(x_n + jh), y''_{n+j}, j = 0(1)4$ and $y''(x_n + 4h)$. Also, assume that $y(x_{n+j}) = y_{n+j}$, $y'(x_{n+j}) = f_{n+j}$, $y''(x_{n+4}) = g_{n+4}$. Then by substituting these into method in equation (24) and after simple algebraic simplification, we obtain

$$LTE = y(x_{n+4}) - y_{n+4}$$
$$= C_{10}h^{10}\left(\omega^2 y^{(8)}(x_n) + y^{(10)}(x_n)\right) + O\left(h^{(11)}\right)$$

Consequently, the Local Truncation Errors (LTE) of BITM are respectively obtained as

$$\mathrm{LTE}=\begin{bmatrix}\frac{557h^{10}}{24444000}\left(y^{(10)}(x_n)+\omega^2y^{(8)}(x_n)\right)+O(h^{11})\\ \frac{199h^{10}}{18333000}\left(y^{(10)}(x_n)+\omega^2y^{(8)}(x_n)\right)+O(h^{11})\\ \frac{733h^{10}}{24444000}\left(y^{(10)}(x_n)+\omega^2y^{(8)}(x_n)\right)+O(h^{11})\\ \frac{4h^{10}}{254625}\left(y^{(10)}(x_n)+\omega^2y^{(8)}(x_n)\right)+O(h^{11})\end{bmatrix} \quad (28)$$

From equation (28), the order of BITM is $p=(9,9,9,9)^T$ with error constants $C_{10}=\left(\frac{557}{24444000},\frac{199}{18333000},\frac{733}{24444000},\frac{4}{254625}\right)^T$. Also, following the definition of [13] and [23], a numerical method is consistent if its order is greater than one. We therefore remark that BITM is consistent. We also remark that that the order obtained in theorems 2 and 3 are in agreement.

3.2 Stability of BITM

Following [24], the BITM is represented in a block matrix form as

$$(A^{(1)}\otimes I)Y_{\mu+1}=(A^{(0)}\otimes I)Y_\mu+h(B^{(1)}\otimes I)F_{\mu+1}+h(B^{(0)}\otimes I)F_\mu+h^2(D^{(1)}\otimes I)G_{\mu+1} \quad (29)$$

where $Y_{\mu+1}=(y_{n+1},y_{n+2},y_{n+3},y_{n+4})^T, Y_\mu=(y_{n-3},y_{n-2}\ y_{n-1},y_n)^T$,

$F_{\mu+1}=(f_{n+1},f_{n+2},f_{n+3},f_{n+4})^T, F_\mu=(f_{n-3},f_{n-2},f_{n-1},f_n)^T,\ G_{\mu+1}=$ $(g_{n+1},g_{n+2},g_{n+3},g_{n+4})^T$ I is an identity matrix, $\otimes$ is the kronecker product of matrices and $A^{(0)},A^{(1)},B^{(0)},B^{(1)},D^{(1)}$ are 4×4 matrices specified as follows

$$A^{(1)}=\begin{bmatrix}\overline{\alpha_{1,1}} & \overline{\alpha_{2,1}} & \overline{\alpha_{3,1}} & 0\\ \overline{\alpha_{1,2}} & \overline{\alpha_{2,2}} & \overline{\alpha_{3,2}} & 0\\ \overline{\alpha_{1,3}} & \overline{\alpha_{2,3}} & \overline{\alpha_{3,3}} & 0\\ \alpha_1 & \alpha_2 & \alpha_3 & 1\end{bmatrix},\quad A^{(0)}=\begin{bmatrix}0&0&0&\overline{\alpha_{0,1}}\\0&0&0&\overline{\alpha_{0,2}}\\0&0&0&\overline{\alpha_{0,3}}\\0&0&0&\alpha_0\end{bmatrix},\quad B^{(1)}=\begin{bmatrix}\overline{\beta_{1,1}} & \overline{\beta_{2,1}} & \overline{\beta_{3,1}} & \overline{\beta_{4,1}}\\ \overline{\beta_{1,2}} & \overline{\beta_{2,2}} & \overline{\beta_{3,2}} & \overline{\beta_{4,2}}\\ \overline{\beta_{1,3}} & \overline{\beta_{2,3}} & \overline{\beta_{3,3}} & \overline{\beta_{4,3}}\\ \beta_1 & \beta_2 & \beta_3 & \beta_4\end{bmatrix},$$

$$B^{(0)}=\begin{bmatrix}0&0&0&\overline{\beta_{0,1}}\\0&0&0&\overline{\beta_{0,2}}\\0&0&0&\overline{\beta_{0,3}}\\0&0&0&\beta_0\end{bmatrix},\quad D^{(1)}=\begin{bmatrix}0&0&0&\overline{\gamma_{4,1}}\\0&0&0&\overline{\gamma_{4,2}}\\0&0&0&\overline{\gamma_{4,3}}\\0&0&0&\gamma_4\end{bmatrix}$$

3.3 Zero Stability

According to [13] and [23], a numerical method is zero stable if the roots of the first characteristic polynomial have modulus less than or equal to one and those of modulus one are simple. i.e. $\rho(R) = \det\left[RA^{(1)} - A^{(0)}\right] = 0$ and $|R_i| \le 1$.

Theorem 4

BITM is zero stable.

Proof

From the normalized first characteristic polynomial of BITM, we have in canonical form that

$$RA^{(1)} - A^{(0)} = \begin{bmatrix} R\overline{\alpha_{1,1}} & R\overline{\alpha_{2,1}} & R\overline{\alpha_{3,1}} & -\overline{\alpha_{0,1}} \\ R\overline{\alpha_{1,2}} & R\overline{\alpha_{2,2}} & R\overline{\alpha_{3,2}} & -\overline{\alpha_{0,2}} \\ R\overline{\alpha_{1,3}} & R\overline{\alpha_{2,3}} & R\overline{\alpha_{3,3}} & -\overline{\alpha_{0,3}} \\ R\alpha_1 & R\alpha_2 & R\alpha_3 & R-\alpha_0 \end{bmatrix}$$

so that $\rho(R) = \det\left[RA^{(1)} - A^{(0)}\right] = 0 \implies R^3(R+1) = 0.$

Consequently, $|R| = 0,0,0$ or $|R| = 1$. Hence the proof.

3.4 Linear Stability and Region of Absolute Stability of BITM

Applying the block method to the test equations $y' = \lambda y$ and $y'' = \lambda^2 y$ and letting $z = \lambda h$ yields $Y_{w+1} = \mathrm{K}(z,u)Y_w$, where $\mathrm{K}(z,u) = \frac{A^{(0)}+zB^{(0)}}{A^{(1)}-zB^{(0)}-z^2D^{(1)}}$. The matrix $\mathrm{K}(z,u)$ for BITM has eigenvalues given by $(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = (0,0,0,\varphi_4)$, where $\varphi_4(z,u) = \frac{p_4(z,u)}{q_4(z,u)}$ is called the stability function which is used to determine the stability region of the BITM. We give the following definition as contained in the literature.

Definition 3.1 [17]. A region of stability is a region in the $z-u$ plane throughout which the spectral radius $\left|\rho\left(\mathrm{K}(z,u)\right)\right| \le 1$.

The $z-u$ plot constructed for BITM is presented in Figure 1

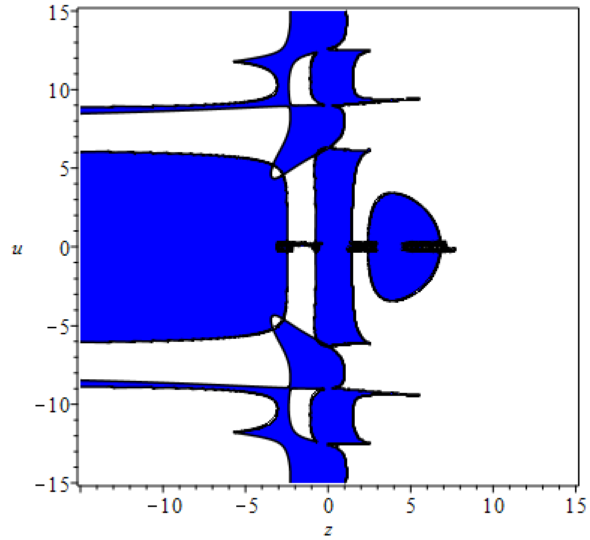


Fig.1. $z-u$ plot for BITM

4. IMPLEMENTATION OF BITM

The application of the BITM with angular frequency ω, on equation (1) in the interval of integration $[x_0, x_N]$ is partitioned with $N \in \mathbb{Z}, N > 0$ for fixed step length such that $h = \frac{b-a}{N}$, and the number of blocks for BITM is $\Lambda = \frac{N}{4}$. To obtain the first block using equation (28) with $n = 0$ and $\mu = 0$, the first numerical results $\{y_1, y_2, y_3, y_4\}$ are simultaneously produced over the subinterval $[x_0, x_4] = [x_0, x_0 + 4h]$, since y_0 and y_0' are known from the IVP under consideration. For the second block, $n = 4$ and $\mu = 1$, the values of $(y_5, y_6, y_7, y_8)^T$ are simultaneously obtained over the subinterval $[x_4, x_8] = [x_0 + 4h, x_0 + 8h]$ as y_4 and y_4' are known from the previous block. This procedure is continued for $n = 8, \cdots, N-4$ and $\mu = 2, \cdots, \Lambda$ respectively to obtain the numerical solution to equation (1) on the entire interval of integration over non overlapping subinterval $\{[x_0, x_4], [x_4, x_8], \cdots [x_{N-4}, x_N]\}$ which makes the BITM self-starting and does not suffer the disadvantage of predictor-corrector modes.

We note that the implementation of BITM was done with the aid of written codes in Maple 2016.2 software enhanced by the feature of *fsolve* for nonlinear problems, and executed on Windows 10 operating system.

4.1 Numerical Examples

In this section, we present a number of nonlinear periodic problem to illustrate the accuracy and efficiency of the BITM. We have calculated the maximum absolute error of the approximate solution on $[x_0, x_N]$ as $Err = \max|y(x) - y|$, the rate of convergence (ROC) is calculated as $\text{ROC} = \log_2(\frac{Err^h}{Err^{2h}})$, where Err^h is the error obtained using the step size h, and the computational efficiency is obtained by plotting the logarithm of the maximum error $(\log(Err))$ versus number of function evaluations (NFEs). It is worthy to note that the fitting frequency used in our implementations are obtained from the problem reference from the literature. However, where such is not stated, the computational frequency is estimated as described in [26] and [27]. Although the frequency choice technique studied by [7], [4] and [5], [28] and [29] and [30] can be explored.

Example 1: Nonlinear Strehmel-Weiner Problem

Consider the nonlinear second order IVP in the interval $0 \leq t \leq 10$ given by

$$y_1''(t) = (y_1(t) - y_2(t))^3 + 6368y_1(t) - 6384y_2(t) + 42\cos(10t), \quad y_1(0) = 0.5, y_1'(0) = 0$$

$$y_1''(t) = -(y_1(t) - y_2(t))^3 + 12768y_1(t) - 12784y_2(t) + 42\cos(10t), \quad y_2(0) = 0.5, y_2'(0) = 0$$

with solution in closed form given by $y_1(t) = y_2(t) = \cos(4t) - \frac{\cos(10t)}{2}$.

This problem was considered and solved for with a sixth order Symmetric Boundary Value Method (SBVM) in [31], a fifth order Block Hybrid Trigonometrically-Fitted Method in [32], Trigonometric Implicit Runge-Kutta Methods (TIRKM) in [15] and Trigonometrically-Fitted Third Derivative Runge-Kutta Nyström Method (TTRKNM) in [17]. This problem is selected to establish the efficiency of BITM on Strehmel-Weiner problem. The results obtained using BITM with $\omega = 4$ are shown in Figures 2 as compared to the aforementioned methods while the accuracy of BITM is plotted in Figures 3a and 3b respectively.

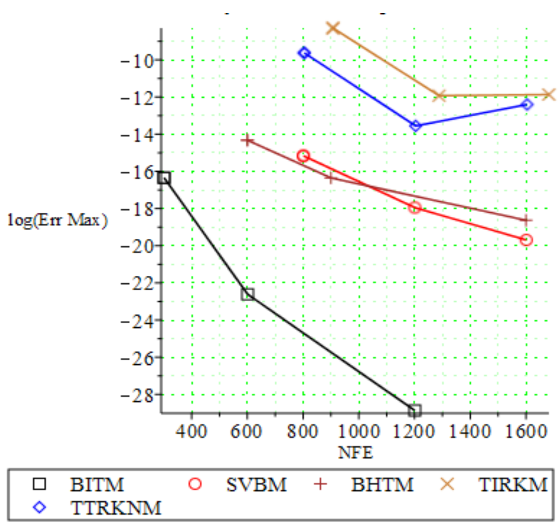


Fig.2. Efficiency curve for Example 1

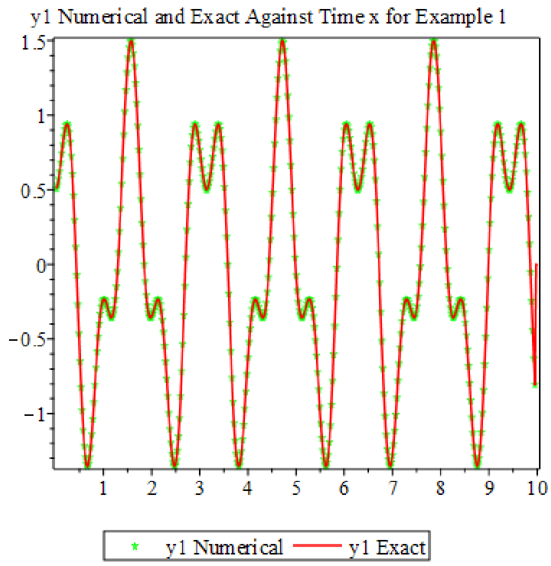


Fig.3a. Accuracy curve for Example 1

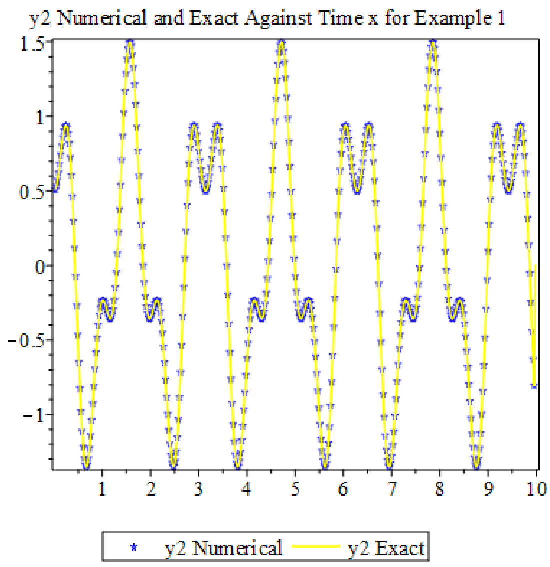


Fig.3b. Accuracy curve for Example 1

We note from Figure 2 that Although SVBM and TTRKNM are direct numerical integrators (without reducing to system of first order IVP) for this problem, BITM has least maximum errors and uses fewer number of function evaluation and consequently a more accurate and more efficient integrator for this problem.

Example 2: Non-Linear Perturbed Systems

As our second test, we consider the nonlinear perturbed system on the range $[0\,,10]$ with $\epsilon = 10^{-3}$.

$$y_1'' = \epsilon\varphi_1(x) - 25y_1 - \epsilon(y_1^2 + y_2^2) \qquad y_1(0) = 1 \;\; , \;\; y_1'(0) = 0$$
$$y_2'' = \epsilon\varphi_2(x) - 25y_2 - \epsilon(y_1^2 + y_2^2) \qquad y_2(0) = \epsilon \;\; , \;\; y_2'(0) = 5$$

where

$$\varphi_1(x) = 1 + \epsilon^2 + 2\epsilon \sin(5x + x^2) + 2\cos(x^2) + (25 - 4x^2)\sin(x^2)$$
$$\varphi_2(x) = 1 + \epsilon^2 + 2\epsilon \sin(5x + x^2) - 2\sin(x^2) + (25 - 4x^2)\cos(x^2)$$

The exact solution is given by $y_1(x) = \cos(5x) + \epsilon \sin(x^2)$, $y_2(x) = \sin(5x) + \epsilon \cos(x^2)$ which represents a periodic motion of constant frequency with small perturbation of variable frequency. This problem was selected to show the performance of BITM on a nonlinear perturbed system. Thus, we choose $\omega = 5$ as the fitting frequency, and the numerical results of the maximum global errors of BITM were compared with a fifth order Trigonometrically-Fitted Adapted Runge-Kutta-Nyström (TFARKN) methods in [35], a fifth order trigonometrically-fitted explicit method (TRI5) in [33], a sixth order hybrid method with dissipation order seven (DIS6) and sixth order hybrid method with Zero dissipative (ZER6) both in [36] as presented Figure 4. The accuracy of BITM is presented in Figures 5a and 5b respectively.

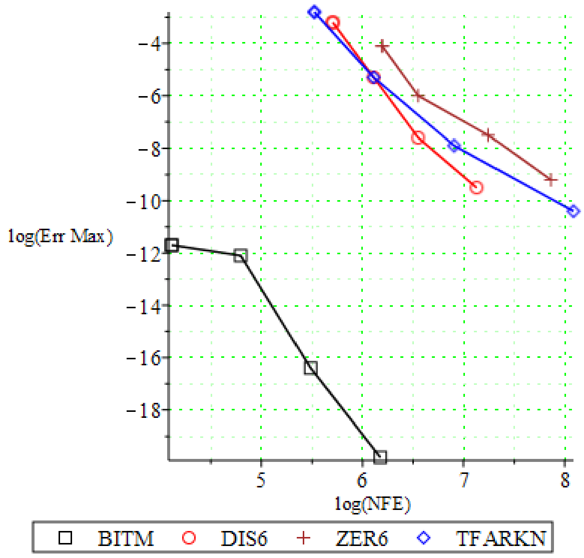


Fig.4. Efficiency curve for Example 2

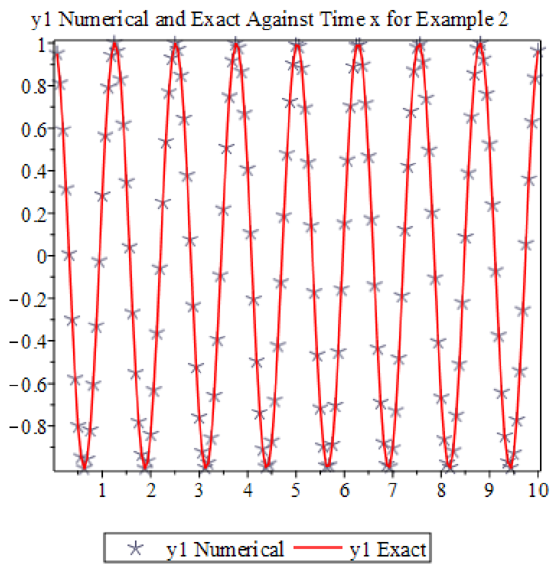


Fig.5a. Accuracy curve for Example 2

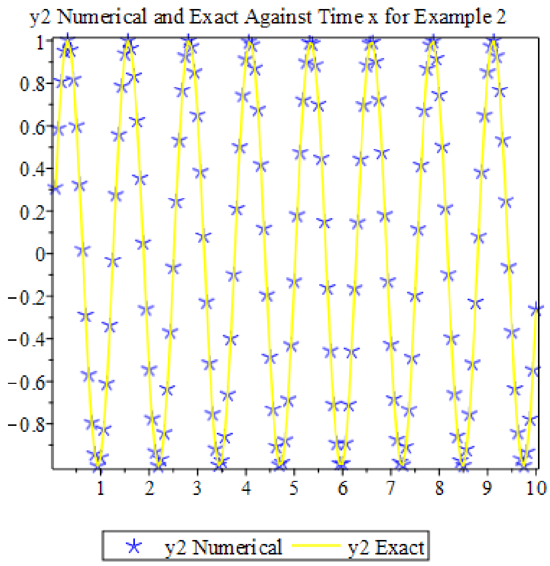


Fig.5b. Accuracy curve for Example 2

Details of the results given in Figure 4 show that the BITM is more efficient than the methods in [35], [33] and [36] respectively.

Example 3: Nonlinear Duffing Equation

We consider the nonlinear Duffing equation forced by a harmonic function given by $y'' + y + y^3 = B\cos(\Omega x)$ whose theoretical solution is unknown. A very accurate approximation of the theoretical solution of this equation is judge by comparison with a Galerkin approximation obtained by [34] given by

$$y(x) = C_1\cos(\Omega x) + C_2\cos(3\Omega x) + C_3\cos(5\Omega x) + C_4\cos(7\Omega x)$$

and the appropriate initial conditions are $y(0) = C_0$ $y'(0) = 0$ where $\Omega = 1.01$, $B = 0.002$, $C_0 = 0.200426728069$ $C_1 = 0.200179477536, C_2 = 0.246946143 \times 10^{-3}$, $C_3 = 0.304016 \times 10^{-6}$, $C_4 = 0.374 \times 10^{-9}$.

This problem has been solved numerically by different researchers in the literature. An explicit eight order method (EM8) was considered in [10], a seventh order hybrid linear multistep method (HLMM) was used in [16] while [11] considered both eighth order block third derivative formulae (BTDF8) and tenth order block third derivative formulae (BTDF10) respectively all in the interval $\left[0, \frac{20.5\pi}{1.01}\right]$. The BITM is specifically compared with BTDF10 because they have the same number of steps. The efficiency curve and the accuracy curve of BITM are presented in Figures 6 and 7 respectively.

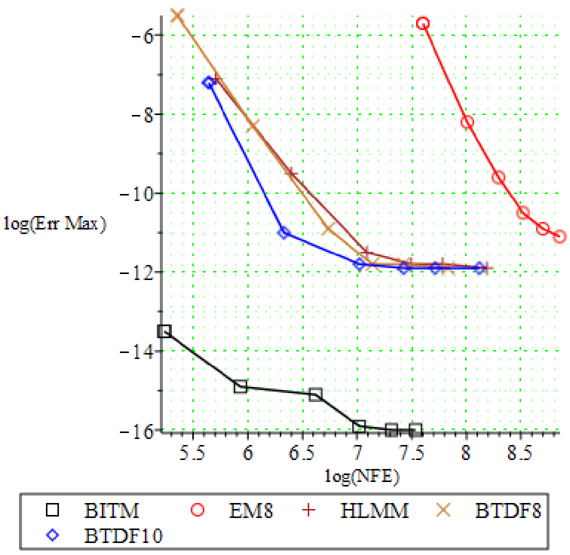


Fig.6. Efficiency curve for Example 3

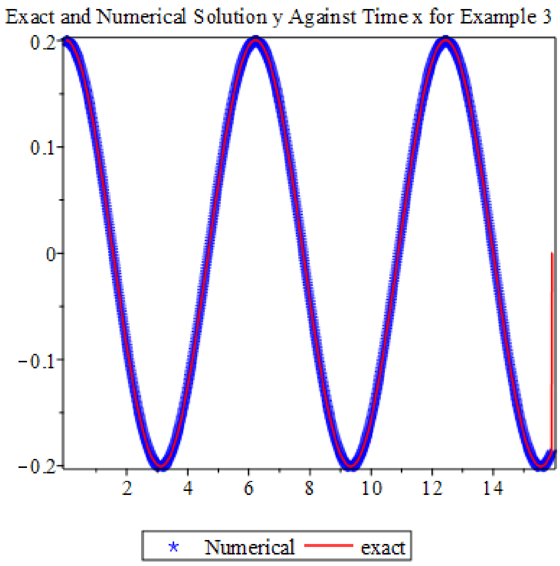


Fig.7. Accuracy curve for Example 3

As revealed by the results in Figure 6, the BITM is a more efficient integrator for nonlinear duffing equation within the considered interval of integration than other methods it compares with which are direct integrators for this problem even a higher order method BTDF10 in [11].

Example 4: Undamped Duffing Equation

Consider the periodically forced nonlinear IVP

$$\begin{cases} y'' = (\cos(t) + \epsilon \sin(10t))^3 - 99\epsilon \sin(10t) - y^3 - y, & 0 \leq t \leq 1000 \\ y(0) = 1, \ \ y'(0) = 10\epsilon \end{cases}$$

with $\epsilon = 10^{-10}$ and whose analytic solution $y(t) = \cos(t) + \epsilon \sin(10t)$ describes a periodic motion of low frequency with a small perturbation of high frequency. In this problem, $\omega = 1$ is selected and the numerical results of BITM in comparison with Block Hybrid Trigonometrically-Fitted Method (BHTM), Trigonometrically-Fitted Adapted Runge-Kutta-Nyström (TFARKN) and Exponentially Fitted Runge-Kutta-Nyström (EFRKN) in [32], [35] and [12] respectively are displayed in Figure 8 while the results of BITM in comparison with the exact solution is presented in Figure 9.

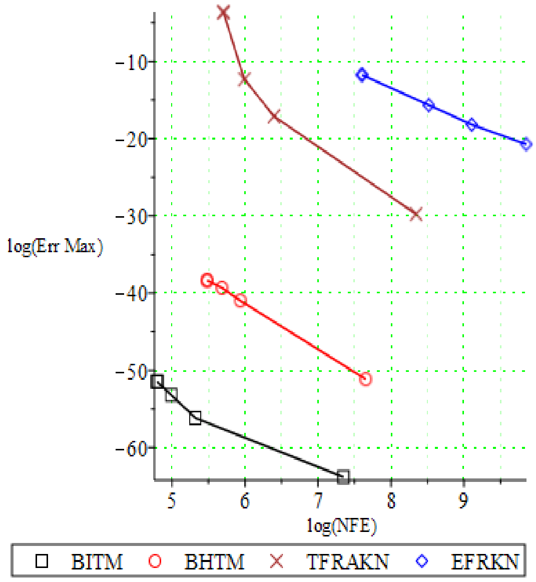


Fig.8. Efficiency curve for Example 4

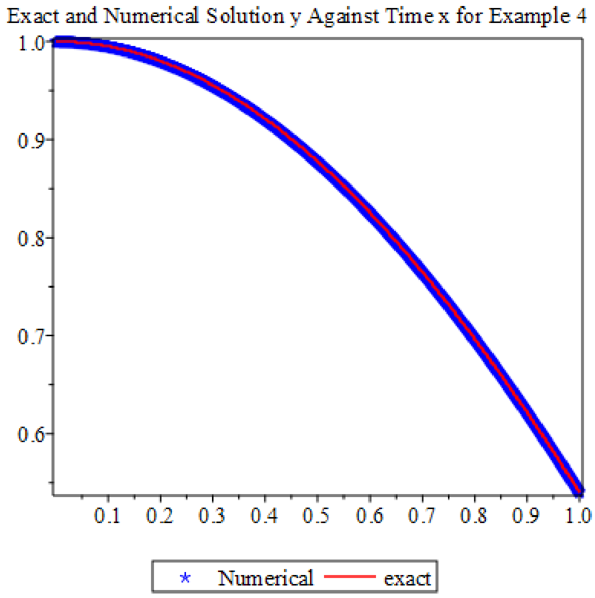


Fig.9. Accuracy curve for Example 4

As expected, the BITM is more efficient than the respective methods in [32], [35] and [12] since it is of higher order as shown in Figure 8.

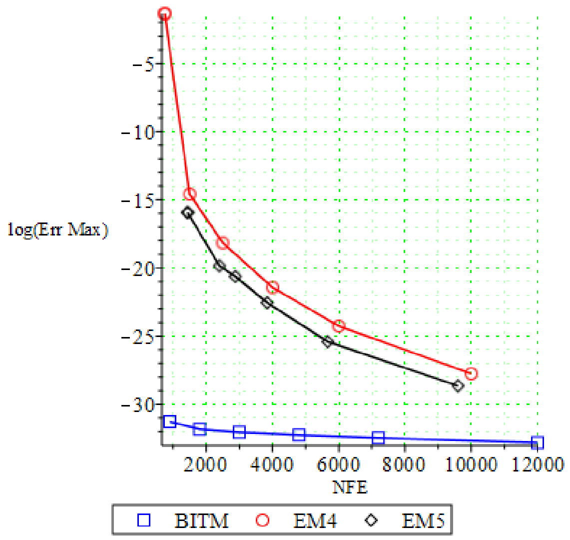
Example 5: Kepler's Problem

As our fifth test, we consider the following system of coupled differential equations which is well known as the two body problem:

$$y_1''(x) = -\frac{y_1}{r^3}, \quad y_1(0) = 1 - e \, , y_1'(0) = 0$$

$$y_2''(x) = -\frac{y_2}{r^3}\, , \quad y_2(0) = 0 \, , y_2'(0) = \sqrt{\frac{1+e}{1-e}}$$

where $r = \sqrt{y_1^2 + y_2^2}$, $x \in [0\, , 50\pi]$, $e(\, 0 \le e < 1)$ is an eccenticity and whose analytical solution is given by $y_1(x) = \cos(k) - e$, $y_2(x) = \sqrt{1-e^2}\sin(k)$, where k is the solution of the Kepler's equation $k = x + e\sin(k)$. For any value of e, the solution of this problem is periodic with period 2π and when $e = 0$, the problem is purely sinusoidal [25]. It is worthy of mentioning that this problem has widely been considered in the literature using methods with lower order. However, we choose to compare BITM with Enright Second Derivative Method of order six and seven (EM4 and EM5) respectively in [30] because they are all variants of Backward Differentiation Formula (BDF), both EM4 and BITM are of the same step size (EM4 is of Adams type while BITM is of the general linear multistep), they required the problem being reduce to system of first order IVP cum they are implemented in block by block fashion. Thus we integrate the problem with $e = 0.005$ and $\omega = 1$, and the results obtained are displayed in Figures 10, 11a and 11b respectively.



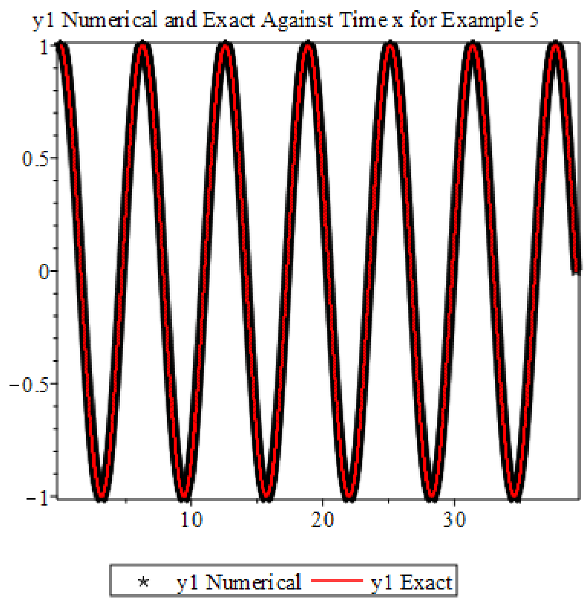


Fig.10. Efficiency curve for Example 5 **Fig.11a.** Accuracy curve for Example 5

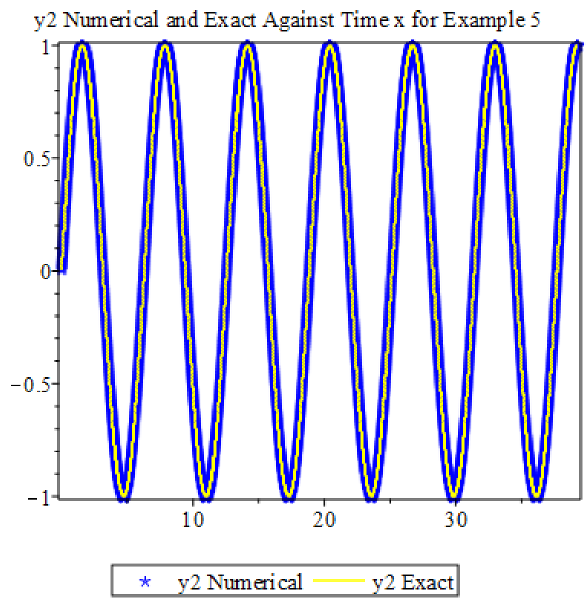


Fig.11b. Accuracy curve for Example 5

Although EM5 is an integrator with 5 steps and uses fewer function evaluation per step compare to the BITM, our method is more accurate. Nevertheless, as shown in Figure 6, the BITM is an efficient integrator for this problem.

5. CONCLUSION

A self-starting, more accurate and more efficient integrator of nonlinear second order IVP with periodic solutions is proposed and applied in this paper. The advantage of BITM in terms of accuracy and efficiency are presented in Figures 2-6.

6. ACKNOWLEDGEMENTS

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