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Research Article

A NEW SYMMETRIC DIFFERENTIAL OPERATOR OF NORMALIZED FUNCTIONS WITH APPLICATIONS IN IMAGE PROCESSING

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ABSTRACT

Recently, a symmetric differential operator (SDO) is attracted to studying in the field of mathematical analysis. A new formal of SDO is presented to generalize some well known differential operators in a complex domain. According to this formulation, we shall exam the boundedness and compactness of this operator in complex spaces, such as Hilbert space and Sobolev space. For this purpose, we suggest new norms to solve the fractional Beltrami equation in the open unit disk. This operator has ability to depict the analytic geometric representation of the solution of second order differential equation utilizing the concept of Schwarzian derivative in the open unit disk. Applications in imagings are given the sequel.

Keywords: fractional calculus; subordination and superordination; differential operator; unit disk; analytic function; subordination; fractional operator; univalent function.

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1. INTRODUCTION

It is well known that the divergence and gradient operators are formulated in the sense of weak derivatives [1]. This property implies a weak solution for various classes of partial differential equations in a complex domain. Such a problem is ill-posed in the meaning of

Hadamard and Sobolev spaces. Therefore, many authors suggested these operators in Hardy spaces. The symmetry of differential operators mentions to the opportunity further down certain conditions of exchanging the order of taking partial derivatives of functions. The most important symmetric operator is Dunkl operator. This type of operators indicated they form an $n \times n$ symmetric matrix. This sometimes is identified as Schwarz's theorem, Young's theorem, or Clairaut's theorem.

The philosophy of distributions (general functions) removes analytic issues with the symmetry. The derivative of an integrable function can always be indicated as a distribution, and symmetry of assorted partial derivatives constantly holds as an equality of distributions. The utilize of official integration by parts to formulate differentiation of distributions sets the symmetry question back onto the examination functions, which are smooth (has derivatives everywhere) and definitely fulfill this symmetry [2-8].

Our aim is to derive a new SDO in the open unit disk and show that its close in some Sobolev spaces based on the conjugate Beltrami equation. Moreover, the existence of conformal weldings can be defined by utilizing the Beltrami equation **Error!** where $z\epsilon\ U$ (the open unit disk) and v is a given complex function in $L^p(U)$ of norm less than 1. This is a process in the geometric function theory by finding univalent functions f and g in the open unit disk and its complement into the extended complex plane.

Also, we have to exam the boundedness and compactness of SDO in Hilbert and Sobolev spaces. For our study, we suggest new norms to solve the fractional Beltrami equation in the open unit disk. This operator has an ability to depict the analytic geometric representation of the solution of second order differential equation utilizing the concept of Schwarzian derivative in the open unit disk. We apply it to recognize the symmetry in leaf images.

2. METHODOLOGY

In this section, we deliver a new SDO in the open unit disk. Let Λ be the class of normalized analytic function formulated by

$$\phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n, \quad z \in U = \{z: |z| < 1\}. \tag{1}$$

For a function $\phi \in \Lambda$, we present a new differential operator in U

$$\begin{aligned}
&\Delta_{\alpha}^{0}\phi(z) = \phi(z)\\
&\Delta_{\alpha}^{1}\phi(z) = \left(\frac{\alpha}{\bar{\alpha}}\right) z\phi'(z) - \left(1-\frac{\alpha}{\bar{\alpha}}\right) z\phi'(-z)\\
&\vdots\\
&\Delta_{\alpha}^{m}\phi(z) = \Delta_{\alpha}(\Delta_{\alpha}^{m-1}\phi(z))\\
&\qquad = z + \sum_{n=2}^{\infty}\left(n\left(\frac{\alpha}{\bar{\alpha}} - (1-\frac{\alpha}{\bar{\alpha}})(-1)^{n}\right)\right)^{m} \phi_n z^n\\
&\qquad := \phi(z) * \mathfrak{D}(z),
\end{aligned} \tag{2}$$

where α is a complex number with its conjugate satisfying |α|<1 and * is the convolution product. It is clear that when α=0, we have the Sàlàgean's differential operator [2]. We call $\Delta^{m,\alpha}$ the Sàlàgean-difference operator. Moreover, $\Delta^{m,\alpha}$ is a new type of Dunkl operator of in the open unit disk (for recent work see [3-5]).

Dunkl operator describes a major modification of partial derivatives and realizes the commutative law in R^n with a Dunkl parameter κ. In geometry, it attains the reflexive property, which is plotting a set of fixed points in the space. An $L^p(U)$ space can be demarcated as a space of functions in U of Lebesgue integral. More commonly, the functions whose absolute value increased to the power (p) has a finite integral

$$\| \phi \|_{L^p} \equiv \left(\int_U |\phi|^p \, d\mu\right)^{1/p} < \infty.$$

Our work is in the generalized Sobolev space **Error!** with the norm

$$\| \phi \|_{W^{1,p}(U)}^{p} = \| \phi \|_{L^p(U)}^{p} + \| \partial\phi \|_{L^p(U)}^{p} + \| \bar{\partial} f \|_{L^p(U)}^{p}$$

where ∂ and $\bar{\partial}$ respectively denote the derivation operators with respect to complex variables $z = x + iy$ and its conjugate $\bar{z}$. Note that $\partial\bar{\partial} = \bar{\partial}\partial = div(\nabla)$. Moreover, We shall deal with another generalized Sobolev space $W^{2,p}(U)$ with the norm

$$\| \phi \|_{W^{2,p}(U)}^{p} = \| \phi \|_{W^{1,p}(U)}^{p} + \| \partial\phi \|_{W^{1,p}(U)}^{p} + \| \bar{\partial}\phi \|_{W^{1,p}(U)}^{p}.$$

The aim of this effort is to show that the operator D_{κ}^{m} is closed under the generalized Sobolev spaces $W^{1,p}(U)$ and $W^{2,p}(U)$, employing the Beltrami equation (B),

$$\bar{\partial}\phi = \nu\partial\phi, \quad |\nu| < 1, \phi \in \Lambda. \tag{3}$$

And the conjugate Beltrami equation (CBE)

$$\bar{\partial}\phi = \nu\bar{\partial}\phi, \quad |\nu| < 1, \phi \in \Lambda. \tag{4}$$

Based on (2), we have the generalized BE

$$\bar{\partial}(\Delta_\alpha^m \phi(z)) = \nu\partial(\Delta_\alpha^m \phi(z)), \quad |\nu| < 1, \phi \in \Lambda. \tag{5}$$

And the conjugate Beltrami equation (CBE)

$$\bar{\partial}(\Delta_\alpha^m \phi(z)) = \nu\bar{\partial}(\Delta_\alpha^m \phi(z)), \quad |\nu| < 1, \phi \in \Lambda. \tag{6}$$

Our aim is to investigate the properties of the solution of (5) and (6) using the properties of $\Delta_\alpha^m \phi(z)$ in different spaces.

3. RESULT AND DISCUSSION

Here, our discussion is based on the generalized Hilbert space in term of Eq.(3) as follows:

$$\| \phi \|_{\mathbb{H}} = \| \phi \|_{L^2(U)} + \| \partial f \|_{L^2(U)},$$

where $\partial = '$ satisfies the norm

$$\| \partial\phi \|_{L^2(U)} = \left(\int_U |\partial f(z)|^2 d\mu(z)\right)^{1/2} < \infty.$$

Totally, we have

$$\| \phi \|_{\mathbb{H}} = \left(\int_U |\phi(z)|^2 d\mu(z)\right)^{1/2} + \left(\int_U |\partial\phi(z)|^2 d\mu(z)\right)^{1/2} < \infty.$$

We illustrate our results.

Theorem 1 *Let $\varphi \in \Lambda$. Then $\Delta^{m,\alpha}: H \to H$ is compact.*

Proof.

By applying properties of the convolution product we obtain

$$\begin{aligned}
\| \Delta_\alpha^m \phi \|_{\mathbb{H}} &= (\int_U |\Delta_\alpha^m \phi(z)|^2 d\mu(z))^{1/2} + (\int_U |\partial(\Delta_\alpha^m \phi(z))|^2 d\mu(z))^{1/2} \\
&= (\int_U |\mathfrak{D} * \phi(z)|^2 d\mu(z))^{1/2} + (\int_U |\partial(\mathfrak{D} * \phi(z))|^2 d\mu(z))^{1/2} \\
&= (\int_U |\mathfrak{D} * \phi(z)|^2 d\mu(z))^{1/2} + (\int_U |(\mathfrak{D}/z) * (\phi(z))'|^2 d\mu(z))^{1/2} \\
&= (\int_U |\mathfrak{D} * \phi(z)|^2 d\mu(z))^{1/2} + (\int_U |(\mathfrak{D}/z * \phi'(z))|^2 d\mu(z))^{1/2} \\
&\leq (\int_U |\mathfrak{D}|^2 d\mu(z))^{1/2} * (\int_U |\phi(z)|^2 d\mu(z))^{1/2} \\
&+ (\int_U |\mathfrak{D}/z|^2 d\mu(z))^{1/2} * (\int_U |(\phi(z))'|^2 d\mu(z))^{1/2} \\
&\leq C(\kappa, m)[(\int_U |\phi(z)|^2 d\mu(z))^{1/2} + (\int_U |\partial\phi(z)|^2 d\mu(z))^{1/2}],
\end{aligned}$$

where

$$C_\alpha := \max_{|z|<1}\{(\int_U |\mathfrak{D}|^2 d\mu(z))^{1/2}, (\int_U |\mathfrak{D}/z|^2 d\mu(z))^{1/2}\}.$$

By taking the supremum for the first term over U yields

$$\| \Delta_\alpha^m \phi \|_{\mathbb{H}} \leq C_\alpha \| \phi \|_{\mathbb{H}}.$$

Thus, Δ_α^m is bounded in $\mathbb{H}$. We proceed to show that Δ_α^m is compact. Let $(\phi_n)_{n\in\mathbb{N}}$ be a equence in $\mathbb{H}$ and that $\phi_n \to 0$ uniformly on U as $n \to \infty$. A calculation implies that

$$\begin{aligned}
\| \Delta_\alpha^m f_n \|_{\mathbb{H}} &= (\int_U |\Delta_\alpha^m f_n(z)|^2 d\mu(z))^{1/2} + (\int_U |\partial(\Delta_\alpha^m f_n(z))|^2 d\mu(z))^{1/2} \\
&= (\int_U |\mathfrak{D} * \phi_n(z)|^2 d\mu(z))^{1/2} + (\int_U |\partial(\mathfrak{D} * \phi_n(z))|^2 d\mu(z))^{1/2} \\
&= (\int_U |\mathfrak{D} * \phi_n(z)|^2 d\mu(z))^{1/2} + (\int_U |(\mathfrak{D} * \phi_n(z))'|^2 d\mu(z))^{1/2} \\
&= (\int_U |\mathfrak{D} * \phi_n(z)|^2 d\mu(z))^{1/2} + (\int_U |(\mathfrak{D}/z) * (\phi'_n(z))|^2 d\mu(z))^{1/2} \\
&\leq (\int_U |\mathfrak{D}|^2 d\mu(z))^{1/2} * (\int_U |\phi_n(z)|^2 d\mu(z))^{1/2} \\
&+ (\int_U |\mathfrak{D}/z|^2 d\mu(z))^{1/2} * (\int_U |(\phi_n(z))'|^2 d\mu(z))^{1/2} \\
&\leq C_\alpha[(\int_U |\phi_n(z)|^2 d\mu(z))^{1/2} + (\int_U |\partial\phi_n(z)|^2 d\mu(z))^{1/2}] \\
&\leq C_\alpha \| \phi_n \|_{\mathbb{H}}.
\end{aligned}$$

Since for $\phi_n \to 0$ on U we have $\| \phi_n \|_{\mathbb{H}} \to 0$, with ε (arbitrary small positive number), by consuming $n \to \infty$ in the last conclusion, we receive that $\lim_{n\to\infty} \| \Delta_\alpha^m \phi_n \| = 0$. Thus, Δ_α^m is compact.

This completes the proof.

Next we discuss the concept of Schwarzian derivative (SD). SD plays a significant role in the theory of univalent functions and conformal mapping. SD of a holomorphic function ϕ of a complex variable z is formulated by [6]

$$(S\phi)(z) = \left(\frac{\phi''(z)}{\phi'(z)}\right) - \frac{1}{2}(\phi''(z)\phi'(z))^2 = \frac{\phi'''(z)}{\phi'(z)} - \frac{3}{2}(\phi''(z)\phi'(z))^2$$

Moreover, SD has an essential communication with a second-order linear ordinary differential equation in the complex plane as follows: let $\phi_1(z)$ and $\phi_2(z)$ be two linearly independent holomorphic solutions of

$$\frac{d^2\phi}{dz^2} + R(z)\phi(z) = 0. \tag{7}$$

Then the ratio $\rho(z)=\phi_1(z)/\phi_2(z)$ achieves $(S\rho)(z)=2R(z)$ over the domain on which $\phi_1(z)$ and $\phi_2(z)$ are formulated, and $\phi_2(z)\neq 0$. The converse is also correct: if such a ρ exists, and it is holomorphic on a simply connected domain, then two solutions ϕ_1 and ϕ_2 can be realized which are unique up to a common scale factor. A sufficient condition for univalence is $|S(\phi)|\leq 2$ (see [8]). In this section, we illustrate our results.

Theorem 2 Let $\phi\in\Lambda$. *Then is compact.*

Proof. By the definition of $\mathfrak{D}(z)$, we have

$$\Delta(z) := \frac{(\mathfrak{D}(z)/z)-1}{z} = \sum_{n=2}^{\infty}\left(n\left(\frac{\alpha}{\bar{\alpha}} - (1-\frac{\alpha}{\bar{\alpha}})(-1)^n\right)\right)^m z^{n-2}$$

By applying properties of the convolution product we obtain

$$\begin{aligned}
&\| \Delta_\alpha^m f \|_{\mathbb{W}^{2,2}(U)}\\
&= (\textstyle\int_U |\Delta_\alpha^m\phi(z)|^2 d\mu(z))^{1/2} + (\int_U |(\Delta_\alpha^m\phi(z))'|^2 d\mu(z))^{1/2} + (\int_U |(\Delta_\alpha^m\phi(z))''|^2 d\mu(z))^{1/2}\\
&= (\textstyle\int_U |\mathfrak{D} * \phi(z)|^2 d\mu(z))^{1/2} + (\int_U |(\mathfrak{D} * \phi(z))'|^2 d\mu(z))^{1/2} + (\int_U |(\mathfrak{D} * \phi(z))''|^2 d\mu(z))^{1/2}\\
&= (\textstyle\int_U |\mathfrak{D} * \phi(z)|^2 d\mu(z))^{1/2} + (\int_U |(\mathfrak{D}/z) * (\phi(z))'|^2 d\mu(z))^{1/2} + (\int_U |(\Delta * (\phi(z))''|^2 d\mu(z))^{1/2}\\
&\leq (\textstyle\int_U |\mathfrak{D}(z)|^2 d\mu(z))^{1/2} * (\int_U |\phi(z)|^2 d\mu(z))^{1/2} + (\int_U |\mathfrak{D}/z|^2 d\mu(z))^{1/2} * (\int_U |(\phi(z))'|^2 d\mu(z))^{1/2}\\
&+(\textstyle\int_U |\Delta(z)|^2 d\mu(z))^{1/2} * (\int_U |(\phi(z))''|^2 d\mu(z))^{1/2}\\
&\leq K_\alpha[(\textstyle\int_U |\phi(z)|^2 d\mu(z))^{1/2} + (\int_U |\phi'(z)|^2 d\mu(z))^{1/2} + (\int_U |(\phi(z))''|^2 d\mu(z))^{1/2}],
\end{aligned}$$

where

$$K_\alpha := \max_{|z|<1}\{(\int_U |\mathfrak{D}(z)|^2 d\mu(z))^{1/2}, (\int_U |\mathfrak{D}/z|^2 d\mu(z))^{1/2}, (\int_U |\Delta(z)|^2 d\mu(z))^{1/2}\}.$$

Accumulating the supremum on first term over U yields

$$\| \Delta_\alpha^m \phi \|_{\mathbb{W}^{2,2}(U)} \le K_\alpha \| \phi \|_{\mathbb{W}^{2,2}(U)}.$$

Thus, Δ_α^m is bounded in $\mathbb{W}^{2,2}(U)$. We proceed to show that D_κ^m is compact. Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{W}^{2,2}(U)$ and that $\phi_n \to 0$ uniformly on U as $n \to \infty$. A calculation implies that

$$\begin{aligned}
&\| \Delta_\alpha^m \phi_n \|_{\mathbb{W}^{2,2}(U)} \\
&= (\int_U |\Delta_\alpha^m \phi_n(z)|^2 d\mu(z))^{1/2} + (\int_U |(\Delta_\alpha^m \phi_n(z))'|^2 d\mu(z))^{1/2} + (\int_U |(\Delta_\alpha^m \phi_n(z))''|^2 d\mu(z))^{1/2} \\
&= (\int_U |\mathfrak{D} * \phi_n(z)|^2 d\mu(z))^{1/2} + (\int_U |(\mathfrak{D} * \phi_n(z))'|^2 d\mu(z))^{1/2} + (\int_U |(\mathfrak{D} * \phi_n(z))''|^2 d\mu(z))^{1/2} \\
&= (\int_U |\mathfrak{D} * \phi_n(z)|^2 d\mu(z))^{1/2} + (\int_U |(\mathfrak{D}/z) * (\phi_n(z))'|^2 d\mu(z))^{1/2} + (\int_U |(\Delta * (\phi_n(z))''|^2 d\mu(z))^{1/2} \\
&\le (\int_U |\mathfrak{D}(z)|^2 d\mu(z))^{1/2} * (\int_U |\phi_n(z)|^2 d\mu(z))^{1/2} + (\int_U |\mathfrak{D}/z|^2 d\mu(z))^{1/2} * (\int_U |(\phi_n(z))'|^2 d\mu(z))^{1/2} \\
&+ (\int_U |\Delta(z)|^2 d\mu(z))^{1/2} * (\int_U |(\phi_n(z))''|^2 d\mu(z))^{1/2} \\
&\le K_\alpha[(\int_U |\phi_n(z)|^2 d\mu(z))^{1/2} + (\int_U |\phi'_n(z)|^2 d\mu(z))^{1/2} + (\int_U |(\phi_n(z))''|^2 d\mu(z))^{1/2}] \\
&\le K_\alpha \| \phi_n \|_{\mathbb{W}^{2,2}(U)}.
\end{aligned}$$

Consequently, we have

$$\| \phi_n \|_{\mathbb{W}^{2,2}(U)} \to 0,$$

where ε is any small value (positive number), by considering $n \to \infty$ in the last inclusion, we get the limit $lim_{n\to\infty} \| \Delta_\alpha^m \phi_n \| = 0$. Thus, Δ_α^m is compact.

Theorem 3 *Let* $\phi \in \Lambda$ *and* $|\alpha| < 1$. *Then there exist two positive constants* $0 < A \le B < \infty$ *satisfy the inequality*

$$A \| \phi \|_{\mathbb{W}^{2,2}(U)} \le \| \Delta_\alpha^m \phi(z) \|_{\mathbb{W}^{2,2}(U)} \le B \| \phi \|_{\mathbb{W}^{2,2}(U)}.$$

Proof. First, we determine the lower bound of radii of the operator $\mathfrak{D}(z)$. Since

$$\lim_{\alpha\to 0}\left(\frac{\alpha}{\bar{\alpha}}-(1-\frac{\alpha}{\bar{\alpha}})(-1)^n\right)=1,$$

then we have the following computation

$$\begin{aligned}|\mathfrak{D}(z)| &= |z+\textstyle\sum_{n=2}^{\infty}\left(n\left(\frac{\alpha}{\bar{\alpha}}-(1-\frac{\alpha}{\bar{\alpha}})(-1)^n\right)\right)^m z^n| \\ &\geq |(z+\textstyle\sum_{n=2}^{\infty} nz^n) *\ldots* (z+\sum_{n=2}^{\infty} nz^n)| \\ &= |(\frac{z}{(1-z)^2})^m| = |(\frac{z}{(1-z)^2})|^m,\end{aligned}$$

Where

$$K(z)=\frac{z}{(1-z)^2}$$

is the Koebe function. But Koebe function is univalent in U therefore, by letting $z\to 1^+$ the growth theorem implies that

$$|\mathfrak{D}(z)|\geq [\frac{r}{(1+r)^2}]^m.$$

Hence, for $r\to 1$, we obtain

$$|\mathfrak{D}(z)|\geq (\frac{1}{4})^m$$

Now we proceed to determine the lower bound of the radii of $|\mathfrak{D}(z)/z|$.

$$\begin{aligned}|\mathfrak{D}(z)/z| &= |1+\textstyle\sum_{n=2}^{\infty}[\left(\frac{\alpha}{\bar{\alpha}}-(1-\frac{\alpha}{\bar{\alpha}})(-1)^n\right)]^m z^{n-1}| \\ &\geq |(1+\textstyle\sum_{n=2}^{\infty} nz^{n-1}) *\ldots* (1+\sum_{n=2}^{\infty} nz^{n-1})| \\ &= |(\frac{z}{(1-z)^2})'|^m,\end{aligned}$$

By the Distortion Theorem, we have

$$|K'(z)|\geq \frac{1-r}{(1+r)^3}\Rightarrow |\mathfrak{D}(z)/z|\geq [\frac{1-r}{(1+r)^3}]^m.$$

Hence, for $r\to 1$, we get $|\mathfrak{D}(z)/z|\geq 0$. Moreover, we have the inequality

$$
\begin{aligned}
|\Delta(z)| \geq |K''(z)|^m \quad &\geq [\Re(K''(z))]^m = [\Re\{\frac{6z}{(1-z)^4} + \frac{4}{(1-z)^2}\}]^m \\
&= [2\Re\{\frac{2+z}{(1-z)^4}\}]^m \\
&\geq 4^m.
\end{aligned}
$$

A calculation yields

$$
\begin{aligned}
&\| \Delta_\alpha^m \phi \|_{\mathbb{W}^{2,2}(U)} \\
&= (\int_U |\Delta_\alpha^m \phi(z)|^2 d\mu(z))^{1/2} + (\int_U |(\Delta_\alpha^m \phi(z))''|^2 d\mu(z))^{1/2} + (\int_U |(\Delta_\alpha^m \phi(z))'|^2 d\mu(z))^{1/2} \\
&= (\int_U |\mathfrak{D} * \phi(z)|^2 d\mu(z))^{1/2} + (\int_U |(\mathfrak{D} * \phi(z))'|^2 d\mu(z))^{1/2} + (\int_U |(\mathfrak{D} * \phi(z))''|^2 d\mu(z))^{1/2} \\
&= (\int_U |\mathfrak{D} * \phi(z)|^2 d\mu(z))^{1/2} + (\int_U |(\mathfrak{D}/z) * (\phi(z))'|^2 d\mu(z))^{1/2} + (\int_U |(\Delta * (\phi(z))''|^2 d\mu(z))^{1/2} \\
&\geq 4^m [(\int_U |\phi(z)|^2 d\mu(z))^{1/2} + (\int_U |\phi'(z)|^2 d\mu(z))^{1/2} + (\int_U |\phi''(z)|^2 d\mu(z))^{1/2}] \\
&= 4^m \| \phi \|_{\mathbb{W}^{2,2}(U)} := A \| \phi \|_{\mathbb{W}^{2,2}(U)}.
\end{aligned}
$$

Finally, in view of Theorem 2, we let $B := K_\alpha$ such that $A \leq B$. Hence, we obtain that the operator Δ_α^m is a frame operator in the Sobolev space $\mathbb{W}^{2,2}(U)$.

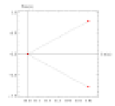
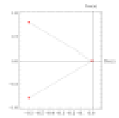
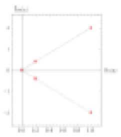
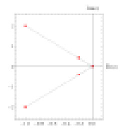
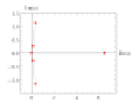
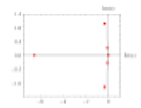
This completes the proof.

4. APPLICATIONS

Image analysis (IA) is a critical study in the area of computer vision, which accords in Symmetry visions. Symmetry is a theory closely related to assembly and uniformity. More accurately, symmetrical rallies can be categorized as covering self-similarities. In 2D images, the self-similarity comes from rigid transformations (such as operators, polynomials and differential and integral equations) that map one part onto another (in calculus is called an injective map and in conformal analysis is called a univalent map).

In 2D- image analysis, recognition of difficulties from their detected outlines is mainly a critical study. It naturally includes a capable illustration of 2D Image analysis with a metric, so that its mathematical structure can be utilized for further analysis. Although the study of 2D simply-connected figures has been subject to the conformal mapping analysis.

Table 1. Examples of SDO: The roots are illustrated in symmetric structures

$\phi(z)$	$\Delta_\alpha^m(z)$	Roots in the complex plane	
$\frac{z}{1-z}$	$\Delta_{0.2}^1 = \frac{0.2z}{(1-z)^2} + \frac{0.8z}{(1+z)^2}$	$0, 0.6 \pm 0.8i$	
-	$\Delta_{0.8}^1 = \frac{0.8z}{(1-z)^2} + \frac{0.2z}{(1+z)^2}$	$0, -0.6 \pm 0.8i$	
$\frac{z}{(1-z)^2}$	$\Delta_{0.2}^1 = -\frac{0.2z(1+z)}{(z-1)^3} + \frac{0.8z(1-z)}{(1+z)^3}$	$0, 0.2 \pm 0.4i, 1 \pm 2i$	
-	$\Delta_{0.8}^1 = -\frac{0.8z(1+z)}{(z-1)^3} + \frac{0.2z(1-z)}{(1+z)^3}$	$0, -0.2 \pm 0.4i, -1 \pm 2i$	
$\frac{z}{(1-z)^3}$	$\Delta_{0.2}^1 = \frac{0.2z(1+2z)}{(1-z)^4} - \frac{0.8z(2z-1)}{(1+z)^4}$	$0, 0.1 \pm 0.2i, 0.3 \pm 1i$	
-	$\Delta_{0.8}^1 = \frac{0.8z(1+2z)}{(1-z)^4} - \frac{0.2z(2z-1)}{(1+z)^4}$	$0, -0.1 \pm 0.2i, -0.3 \pm 1i$	

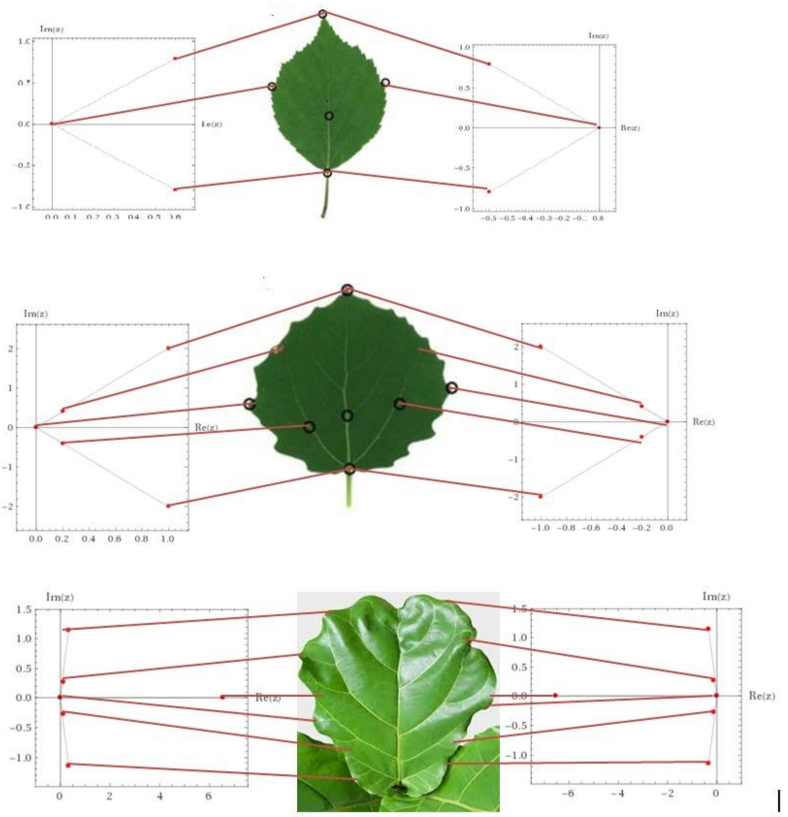


Fig.1. Conformal mapping shape corresponds to the SDO for analytic functions $z/(1-z)^p$

We have illusetrate some exapmles in image processing using the SDO that given in Eq. (2). Our examples based on the analytic function z/(1-z)p, where p=1,2,3 and $|z|<1$ (see Table 1). Fig.1 shows the symmetric shape of leaves by congruent the roots in the opent unit disk with the edges of the leaves. The symmetric shapes are given by taking p=1,p=2 and p=3 respectively. Moreover, our experements shows that the value of $\nu= \alpha/\text{conjugat}\ (\alpha)=0.2$ and $\nu=0.8$ to give a good symmetry. Our algorithm is programing in Mathematica 11.2, the accuracy of this algorithm is given by the ratio between the number of roots (ρ) and the number of angles (λ) surowing the edge of the leaf: $\Upsilon=\rho/\lambda$, with $0<\Upsilon\leq 1$. For example, the accuracy of the first leaf is $\Upsilon=1$, for the second leaf is $\Upsilon=0.5$, while for the last one is $\Upsilon=0.8$. We applied this algorithm is a set of 1000 images; all results are of accuracy $\Upsilon \geqslant 0.5$. Moreover, we can apply this method to recognize a symmetry and insymmetry for all images.

Finally, we conclude our steps in Fig.2

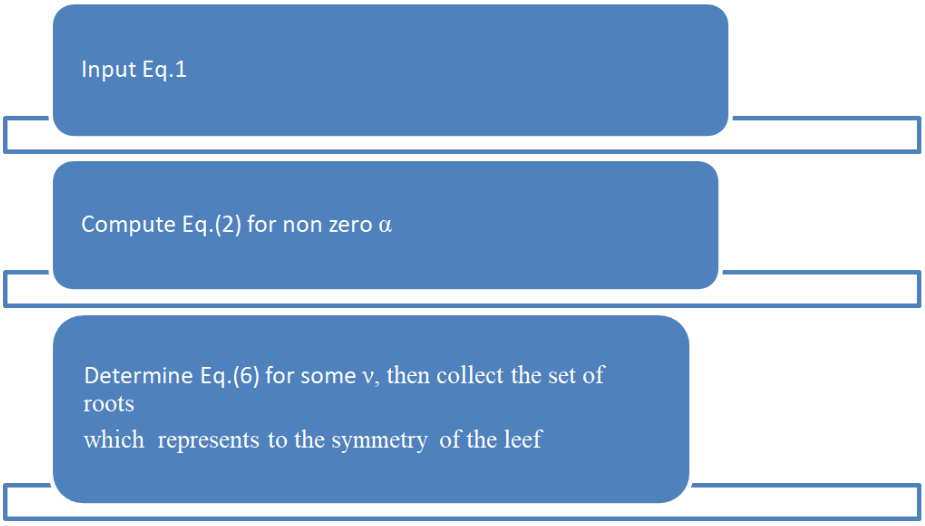


Fig.2. the steps of the algorithm

5. CONCLUSION

Here, we introduced a symmetric differential operator in the open unit disk for normalized functions. The operator satisfied important properties such as a solution of a special Beltrami differential equation. It has been shown that this operation is bounded and compact is some Hilbert spaces. This operator can be extended to other classes of analytic functions such as meromorphic, p-valent and harmonic classes of analytic functions. Examples showed some evaluations of well-known function in the open unit disk. For future work, we suggest to the researchers to use the planned operator in other applications that deal with the symmetry studies.

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